

You may refer to the following languages by name:

$A^n B^n$  is the language  $\{a^n b^n : n \in \mathbf{N}\}$ .

Prime is a language over a unary alphabet  $\{\alpha\}$  where  $\alpha^p \in \text{Prime}$  iff  $p$  is a prime number.

Evenpal over  $\Sigma$  is the language defined as  $\{ww^R : w \in \Sigma^*\}$ .

Pal over  $\Sigma$  is the language defined as  $\{w : w = w^R \in \Sigma^*\}$ .

(Recall that for a string  $w$ ,  $w^R$  is the reverse of  $w$ , and  $L^R = \{w : w^R \in L\}$ .)

1. (2 mark) Either find, if they exist, two non-regular Context-Free languages  $L_1, L_2$  whose intersection is regular, or explain why no such two CF languages exist.
  
  
  
  
  
  
  
  
  
  
2. (6 marks) True or False:
  - (a) \_\_\_ There is a deterministic PDA that accepts the language of palindromes over alphabet  $\Sigma = \{a, b\}$ .
  - (b) \_\_\_ Given a context-free grammar  $G$ , each string in  $L(G)$  has a unique derivation in  $G$ .
  - (c) \_\_\_ It is always the case that if  $L$  is context-free, then so is  $L$ 's complement  $\bar{L}$ .
  
3. (5 marks) Is the class of Context-Free languages closed under reverse? Prove your answer. (Hint: If a language is CF then it has a CF Grammar.)

4. (6 marks) Show how to construct a CF grammar for  $L_1 \cup L_2$ , given that  $L_1$  has the grammar  $(V_1, \Sigma, R_1, S_1)$  and  $L_2$  has the grammar  $(V_2, \Sigma, R_2, S_2)$ . You can assume that  $V_1$  and  $V_2$  are disjoint (their intersection is empty).

5. (12 marks) Give a context-free grammar for each of the following languages:

(a)  $\{a^{i+j+k}b^i c^j d^k : i, j, k \geq 0\}$

(b)  $\{w \in \{(\,), [\,]\}^* : w \text{ is a balanced string of both kinds of parentheses, and square brackets can appear within rounded parentheses but rounded parentheses cannot appear within square brackets}\}$ . E.g.,  $([])[()]$  is a string in the language, but  $[()]$  is not, nor is  $[(])$ .

(c)  $\{w \in \{0, 1, \$\}^* : w \text{ contains exactly one } \$, \text{ and has more } 0\text{'s than } 1\text{'s before the } \$, \text{ and more } 1\text{'s than } 0\text{'s after the } \$\}$

6. (4 marks) What language is generated by the following grammar?

$$S \rightarrow X|SS$$

$$X \rightarrow aXc|B$$

$$B \rightarrow Bb|\epsilon$$

Give a precise description in English, or a precise mathematical description of the language.

7. (5 marks) Give a natural PDA for the language  $L = \{w \in \{a, b, c, \}^* : \#_b(w) \leq \#_a(w)\}$ .

8. (5 marks) Give a Top-Down Parser PDA for the following CF Grammar:

$$S \rightarrow X|SS$$

$$X \rightarrow aXc|B$$

$$B \rightarrow abB|b|\epsilon$$

9. (a) (3 marks) Give an example of a non-regular Context-Free language and an infinite regular language such that their intersection is regular. Give a precise description of the language that is the intersection of the two.

(b) (3 marks) Give an example of a Context-Free language and a Regular language such that their intersection is not regular. Give a precise description of the language that is the intersection of the two.